RESEARCH REPORT 27
1 August 1962
I. E. R. 172-32

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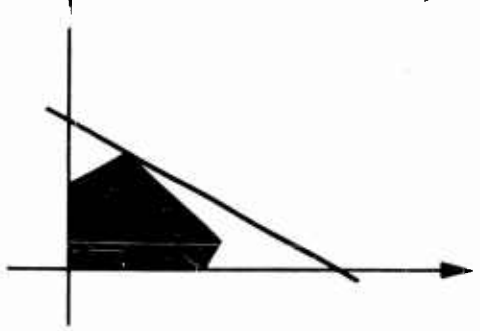
R. Van Slyke and R. Wets



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Richard Van Slyke and Roger Wets
Operations Research Center
University of California, Berkeley

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ON DIAGONALIZATION METHODS IN INTEGER PROGRAMMING

by Richard Van Slyke and Roger Wets

I. Introduction

An important area for improvement in existing integer programming codes is in the easy generation of efficient cutting hyperplanes; in this paper we approach this problem using a triangular canonical form. In the first part we give an algorithm based on Gomory's all-integer integer programming algorithm[2], which constitutes a first step in this direction. This procedure is a practical analog of a deepest cut method discussed in the second part of the paper. In Appendix A, a brief outline and flow diagram for the algorithm are given; finally the algorithm and the deepest cut problem are illustrated by examples in appendices B and C.

We assume that we have at hand an integer program where all the coefficients and constant terms are integers. The functional is to be maximized. We write the problem in a parametric form due to Tucker:

Maximize x_0 subject to x_j integer, $j = 0, \ldots, n$; $x_j \geq 0$, $j = 1, \ldots, n$; and

$$
\begin{aligned}
x_0 &= b_0 + c_0 t_0 + c_1 t_1 + \cdots + c_k t_k \\
x_1 &= b_1 + a_{10} t_0 + a_{11} t_1 + \cdots + a_{1k} t_k \\
&\vdots \\
x_m &= b_m + a_{m0} t_0 + a_{m1} t_1 + \cdots + a_{mk} t_k
\end{aligned}
\tag{1}
$$

It will be convenient to assume that the above system has the property that if $x_0, x_1, \ldots, x_m$ are non-negative integer variables then $t_0, \ldots, t_k$ must also be non-negative integer variables. We can guarantee this, in

general, by adding the trivial relations $x_{i+1} = t_i$, $i = 0, \ldots, k$. In this case the problem will be of the form:

Maximize x_0 subject to x_j non-negative integers, $j = 1, \ldots, n$; and

$$
\begin{aligned}
x_0 &= b_0 + c_0 t_0 + c_1 t_1 + \ldots + c_k t_k \\
x_1 &= t_0 \\
x_2 &= t_1 \\
&\vdots \\
x_{k+1} &= t_k \\
x_{k+2} &= b_{k+2} + a_{k+20} t_0 + a_{k+21} t_1 + \ldots + a_{k+2k} t_k \\
&\vdots \\
x_n &= b_n + a_{n0} t_0 + a_{n1} t_1 + \ldots + a_{nk} t_k
\end{aligned}
\tag{1'}
$$

If in this problem $b_i \geq 0$, $i = 1, 2, \ldots, n$ and $c_j \leq 0$ for $j = 0, 1, \ldots, k$ the solution obtained by setting $t_0 = t_1 = \ldots = t_k = 0$ is optimal.

It can be assumed without loss of generality that for bounded problems the problem is in a dual feasible form, such that $c_j < 0$ for all j. A quite general method for accomplishing this is discussed in a paper by Dantzig, Ford and Fulkerson.[1]

Two Useful Operations

We now describe two basic operations which will frequently be used:

a) The Cut: Suppose

$$x_i = b_i + a_{i0} t_0 + a_{i1} t_1 + \ldots + a_{i\ell} t_\ell + \ldots + a_{ik} t_k \tag{2}$$

is an equation of (1). Let

$$J^+ = \left\{ j : a_{ij} > 0 \right\} \qquad \text{and} \qquad J^- = \left\{ j : a_{ij} \leq 0 \right\} \quad .$$

If x_i is to be a non-negative variable we have the following:

$$0 \leq x_i = b_i + \sum_{j \in J^+} a_{ij} t_j + \sum_{j \in J^-} a_{ij} t_j \quad \leq \quad b_i + \sum_{j \in J^+} a_{ij} t_j \quad .$$

If

$$\tau = b_i + \sum_{j \in J^+} a_{ij} t_j \tag{3}$$

then τ is a non-negative integer valued variable. Using (3) we could eliminate one of the t's appearing in (3), say t_ℓ in (1) in favor of τ. If, moreover, $a_{i\ell} = 1$ making the substitution $t_\ell = \tau - b_i - \sum_{j \in J^+ - \{\ell\}} a_{ij} t_j$ in (1) would leave the tableau in integers.

$$x_0 = b_0 + c_0 t_0 + \cdots + c_\ell \left(\tau - b_i - \sum_{j \in J^+ - \{\ell\}} a_{ij} t_j \right) + \cdots + a_{0k} t_k$$

$$x_1 = b_1 + a_{10} t_0 + \cdots + a_{1\ell} \left(\tau - b_i - \sum_{j \in J^+ - \{\ell\}} a_{ij} t_j \right) + \cdots + a_{1k} t_k$$

$$\vdots \qquad \vdots$$

$$x_n = b_n + a_{n0} t_0 + \cdots + a_{n\ell} \left(\tau - b_i - \sum_{j \in J^+ - \{\ell\}} a_{ij} t_j \right) + \cdots + a_{nk} t_k$$

If $a_{i\ell} > 1$, then

$$(3') \qquad \tau = \frac{b_i^*}{a_{i\ell}} + \sum_{j \in J^+} \frac{a_{ij}^*}{a_{i\ell}} t_j$$

is still a non-negative integer variable where $-b_i^*$ and a_{ij}^* are the smallest integers greater than $-b_i$ and a_{ij}, respectively, which are divisible by $a_{i\ell}$, where $(a_{i\ell}^*/a_{i\ell}) = (a_{i\ell}/a_{i\ell}) = 1$, so if $a_{i\ell} > 1$, we can use (3') and maintain the tableau in integers.

To minimize the notation after the substitution we simply let τ be represented by t_ℓ in the tableau. In general we only use the substitution (3) or (3') when $b_i < 0$ and in this case call it a <u>cut</u> because the restriction $\tau \geq 0$ restricts the values that $t_0, \ldots, t_k$ can take on, e.g., $t_0 = t_1 = \ldots = t_k = 0$ is not a feasible solution.

b) <u>The Column Operation</u>: On the other hand if we define $\tau = \sum_j \alpha_j t_j$, where α_j are non-negative integer parameters and $\alpha_\ell = 1$, and substitute for t_ℓ, we are in fact performing a <u>column operation</u>, because formally the substitution is equivalent to subtracting α_i times the ℓ^{th} column of coefficients of (1') from the i^{th} column of (1') for $i = 0, 1, \ldots, \ell-1, \ell+1, \ldots, k$. We note that this is of the form of a cut with $b_i = 0$. The only thing we have to be careful about in making these substitutions for cuts and column operations is that the optimality condition for the transformed tableau is the same as for the original; more specifically that $x_1, \ldots, x_n \geq 0$ and integer implies that the current $t_0, \ldots, t_k$ are non-negative integer parameters. This is easy to verify remembering that the original tableau had this property and that each new parameter is a non-negative integer weighted sum of $t_0, \ldots, t_k$.

II. The Basic Algorithm

Ralph Gomory's all-integer algorithm concerns itself basically with making "cuts" in a systematic fashion until a transformed tableau of integers is obtained from (1) with $c_j \leq 0$, $(j = 0, \ldots, k)$, $b_i \geq 0$, $(i = 1, \ldots, n)$. The question is how to generate the most "efficient" cuts in some sense. Gomory has suggested[2] that the efficiency may be related to the lexicographic magnitude of the pivot column in the sense that one should try to minimize the magnitude of the pivot column. Whenever we attach adjectives such as "best", "efficient", "deep" to the word "cut" we will always mean it in this way.

1. The Algorithm.

Suppose our problem is in the form:

Maximize x_0 where $x_j \geq 0$, $(j = 1, \ldots, n)$; and x_j are integers $(j = 0, \ldots, n)$ subject to

$$
\begin{array}{lllllll}
x_0 & = b_0 & - & t_0 & & & \\
x_1 & = b_1 & + a_{10}t_0 & - & t_1 & & \\
x_2 & = b_2 & + a_{20}t_0 & + a_{21}t_1 & - & t_2 & \\
\vdots & \vdots & \vdots & \vdots & & & \\
x_{k+1} & = b_{k+1} & + a_{k+1,0}t_0 & + a_{k+1,1}t_1 & + a_{k+1,2}t_2 + \cdots\cdots & - & t_{k+1} \\
x_{k+2} & = b_{k+2} & + a_{k+2,0}t_0 & + a_{k+2,1}t_1 & + a_{k+2,2}t_2 + \cdots\cdots & + a_{k+2,k+1}t_{k+1} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
x_n & = b_n & + a_{n0}t_0 & + a_{n1}t_1 & + a_{n2}t_2 + \cdots & + a_{nk+1}t_{k+1} &
\end{array}
\tag{5}
$$

Then finding efficient cuts is relatively easy since the columns are already ordered lexicographically. We seek a cut $\tau = \beta_i + \sum_j \alpha_j t_j$ for which the

last positive coefficient, which determines the pivot column, is as far to the left as possible. This follows because we always insist on maintaining dual feasibility. It is clear that the more "efficient" a cut, the fewer columns (in general) will be affected by the pivoting. With this criterion in mind we seek to generate the cuts. We will see in the next section that every problem of the form (1') and therefore of the form (1) can be put in the form (5).

We can consider, instead of a single equation, a sum of equations weighted by positive integers to generate the cuts. To each equation say the i^{th}, with a negative constant term, we associate a most efficient cut in the following sense:

(i) If

(a) the coefficient of $t_{k+1}a_{i,k+1}$ is positive we add $a_{i,k+1}$ times the $k+1^{st}$ equation to the i^{th}, if this keeps the constant term $b_i + a_{i,k+1}b_{k+1}$ negative, we go to (ii). If it makes the constant term non-negative then the original equation, the i^{th}, is associated with itself.

(b) $a_{i,k+1}$ is negative or zero we proceed directly to (ii).

(ii) We now have a new linear combination of the t's where the coefficient of t_{k+1} is non-positive. Next we examine the coefficient of t_k which is $a_{i,k} + a_{i,k+1}a_{k+1,k}$ in case (a) or $a_{i,k}$ in case (b). As in (i), if the coefficient is negative or zero we go on to consider t_{k-1}. If this is not the case we add $a_{i,k} + a_{i,k+1}a_{k+1,k}$ times the k^{th} equation if this keeps the constant term negative. If it does not, the current linear combination is associated with the i^{th} equation, and we stop.

(iii) We continue the process until either the constant term goes non-negative and the process terminates or until a positive integer weighted

combination of equations is obtained in which the coefficient of every t_j, $(j \neq 0)$ is non-positive. If one obtains an equation with all coefficients non-positive and with a negative constant term, there is no feasible solution. In this way, we can associate to each equation with a negative constant term another derived equation. We can then choose from the collection of derived equations the one for which the last positive coefficient is farthest to the left. From this equation we generate the cut. We then repeat the whole process with the new tableau.

The convergence properties of this basic algorithm are closely connected to those of Gomory's all-integer algorithm[2]. For completeness we review the proof. Because of the special nature of the tableau there will always be a decrease in the constant term in some row above the highest row with a negative constant term. This, plus the assumption that a feasible solution exists, is enough to guarantee convergence. To see this, we note that for each iteration we add a lexico-negative vector to the constant column, so that the constant column is always decreasing lexicographically. If we indicate by a superscript the iteration numbers for the constant terms, we have for the constant term of the functional:

$$b_0^{(0)} \geq b_0^{(1)} \geq \ldots \geq b_0^{(\nu)} \geq b_0^{(\nu+1)} \geq \ldots \geq b_0^{(\text{optimal, feasible})}$$

where strict inequality can hold at most a finite number of times, i.e., there is a ν_0 such that $b_0^{(\nu)} = b_0^{(\nu_0)}$ for all $\nu \geq \nu_0$. For if this were not so, since $b_0^{(\nu)}$ decreases by an integer each time, we would have strict inequality and the functional value would eventually become less than the functional value of the assumed feasible solution, which is impossible since $x_0 \leq b_0^{(\nu)}$ for every ν. Now from ν_0 on we examine $b_1^{(\nu)} (\nu \geq \nu_0)$. They satisfy the condition $b_1^{(\nu_0+1)} \geq b_1^{(\nu_0+2)} \geq \ldots$. Suppose $b_1^{(\nu)}$ does

not "settle down," i.e., suppose strict inequality does not hold at most a finite number of times, then eventually $b_1^{(\nu)}$ becomes negative. This is a contradiction for the cut generated at this point would cause a strict decrease in b_0 which is impossible since $\nu \geq \nu_0$. So after a finite number of iterations ν_1 $(\geq \nu_0)$ both b_0 and b_1 have "settled down." Suppose $b_0, \ldots, b_{\ell-1}$ have settled down, then if b_ℓ decreases indefinitely it eventually goes negative. Then the cut generated at this point will imply a pivot on some column strictly to the left of the ℓ^{th}. In other words b_i will decrease strictly for some $i < \ell$ which is a contradiction. Since the number of b's is finite, the process terminates in a finite number of steps.

2. Basic Canonical Form.

We show in this section that any problem of form (1) or equivalently (1') can be represented in form (5). Assuming we have the problem already in form (1') we get an equivalent basic canonical form (5) to (1') by subtracting from the first row of the tableau an extra parameter t_{k+1}, and adding two extra equations:

$$x_{n+1} = \quad - t_{k+1}$$

$$x_{n+2} = \quad + t_{k+1}$$

Since x_{n+1} and x_{n+2} are required to be non-negative, x_{n+1}, x_{n+2}, and t_{k+1} must all equal zero in any feasible solution. We have then:

$$
\begin{aligned}
x_0 &= b_0 + c_0t_0 + c_1t_1 + \cdots\cdots + c_kt_k - t_{k+1} \\
x_1 &= t_0 \\
x_2 &= t_1 \\
&\vdots \\
x_{k+1} &= t_k \\
x_{k+2} &= b_{k+2} + a_{k+2,0}t_0 + a_{k+2,1}t_1 + \cdots\cdots + a_{k+2,k}t_k \\
&\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\
x_n &= b_n + a_{n0}t_0 + a_{n1}t_1 + \cdots\cdots + a_{nk}t_k \\
x_{n+1} &= - t_{k+1} \\
x_{n+2} &= + t_{k+1}
\end{aligned} \tag{6}
$$

We then perform the following changes of variables

$$
t_j^{new} = t_{k+1} - (1+c_0)t_0 - \cdots - (1-c_{j-1})t_{j-1} - c_jt_j - c_{j+1}t_{j+1} - \cdots - c_kt_k \tag{7}
$$

for $j = 1, \ldots, k+1$ and

$$
t_0^{new}, \; t_{k+1} - c_0t_0 \cdots\cdots - c_jt_j \cdots\cdots - c_kt_k
$$

The transformations of variables are legitimate because the c_j's are strictly negative; so, all the coefficients in the above equations are positive or zero and the substitutions are column operations. Performing these column operations we obtain the following form:

$$
\begin{aligned}
\text{Max } x_0 &= b_0 - t_0 \\
x_1 &= t_0 - t_1 \\
&\vdots \\
x_{k+1} &= t_k - t_{k+1} \\
x_{k+2} &= b_{k+2} + a_{k+2,0}t_0 + a_{k+2,1}t_1 + \cdots + a_{k+2,k+1}t_{k+1} \\
&\vdots \\
x_{n+2} &= b_{n+2} + a_{n+2,0}t_0 + a_{n+2,1}t_1 + \cdots + a_{n+2,k+1}t_{k+1}
\end{aligned} \tag{8}
$$

where for notational simplicity we do not distinguish typographically the transformed coefficients and parameters of (8) from the old ones in (6). To prove the equivalence of (8) and (6) it suffices to write down the inverse transformation. In fact

$$
\begin{aligned}
t_0 &= t_0^{new} - t_1^{new} \\
t_1 &= t_1^{new} - t_2^{new} \\
&\vdots \\
t_k &= t_k^{new} - t_{k+1}^{new}
\end{aligned}
$$

Tableau (8) is a special case of the desired form (5) with the extra feature that the new a_{ij}'s and b_j's are zero in the lower left-hand corner of the diagonal part of the matrix. The method given above is a completely general method; however the extra variables were added only to supply a coefficient of minus one in the functional. If one already exists, say $c_k = -1$, the same transformations (7) will work with k replaced by k-1.

The algorithm is illustrated by an example in Appendix B.

III. Solving a Special Subproblem for the Deepest Cut

We will now discuss a special problem with interesting properties, which constitutes the basis for the deepest cut algorithm described in this section.

1. The Deepest Cut Problem.

Let us consider the following problem: Find non-negative integers $x_j (j \neq 0)$, t_i, and maximize x_0, subject to:

$$
\begin{aligned}
x_0 &= b_0 - t_0 \\
x_1 &= b_1 + a_{10}t_0 - t_1 \\
x_2 &= b_2 + a_{20}t_0 + a_{21}t_1 - t_2 \\
&\vdots \\
x_k &= b_k + a_{k,0}t_0 + a_{k,1}t_1 + \cdots\cdots + a_{k,k-1}b_{k-1} - t_k \\
x_{k+1} &= -b_{k+1} - a_{k+1,0}t_0 - a_{k+1,1}t_1 - \cdots\cdots - a_{k,k-1}t_{k-1} + t_k
\end{aligned}
\tag{9}
$$

where all the b_j $(j \neq 0)$ and a_{ij} are non-negative integers.

It is not difficult to see that, given this structure, the lexico-minimum of all t vectors generating feasible x's is a solution to the problem. In order to make more apparent some properties of this problem let us rewrite it for $k = 3$.

Find non-negative integers $x_j (j \neq 0)$, t_i and maximize x_0, subject to:

$$
\begin{aligned}
b_0 &= x_0 + t_0 \\
b_1 &= -a_{10}t_0 + x_1 + t_1 \\
b_2 &= -a_{20}t_0 - a_{21}t_1 + x_2 + t_2 \\
b_3 &= -a_{30}t_0 - a_{31}t_1 - a_{32}t_2 + x_3 + t_3 \\
x_4 &= -b_4 - a_{40}t_0 - a_{41}t_1 - a_{42}t_2 + t_3
\end{aligned} \tag{10}
$$

An equivalent statement of this problem is to consider the last equation as the objective and find the lexico-min of all the vectors t which make x_4 non-negative. The solution of this problem is an optimal solution to the initial problem. So, we have then

Objective: $x_4 = -b_4 - a_{40}t_0 - a_{41}t_1 - a_{42}t_2 - t_3$.

The first of the equations of the system (the original functional) is not really a constraint, since t_0 may be increased as much as necessary to achieve primal feasibility. In order to make x_4 feasible (keeping in mind that we are seeking the lexico-min vector t), we will first increase t_3 as much as possible, keeping other t_i variables at zero value. This means, with respect to the simplex method, making t_3 and x_0, x_1, x_2 basic.

If we did not succeed in reaching feasibility, we would want then to increase t_2 keeping t_0 and t_1 at zero level if this can do some good, i.e., if the coefficient of t_2 in the adjusted "objective" (x_4) is greater than zero.

If increasing t_2 as much as possible, (which is equivalent to introducing t_2 in the basis when using the simplex method) does not suffice to make $x_4 \geq 0$, it means that there is no solution with $t_0 = t_1 = 0$. We then continue the process with t_1 and then with t_0 if necessary.

Performing these operations, there will be some j such that making t_j basic will yield a non-negative value for x_4^* (the value of the adjusted objective). This implies that there exist a solution to the problem with $t_0 = t_1 = \ldots = t_{j-1} = 0$, t_j basic and assigning some values to basic $t_{j+1}, t_{j+2}, \ldots, t_k$. But we are seeking the lexico-min vector t and increasing t_j as far as possible is not necessarily required. There may exist a value, say t_j^*, less than the value which is assigned to t_j when we make it basic, which should suffice to make x_4 reach feasibility.

In the case where $k = 3$ (10) let $j = 1$

1) We introduce t_3 in the basis then the "adjusted objective" becomes:

$$(11) \quad x_4^* = x_4 + x_3 = (b_3 - b_4) + (a_{30} - a_{40})t_0 + (a_{31} - a_{41})t_1 + (a_{32} - a_{42})t_2 - x_3$$

2) Assume $b_3 - b_4 < 0$

If $a_{32} - a_{42}$ is > 0, we then consider increasing the value of t_2 and the new "adjusted objective" will be the following linear combination

$$x_4^* = (a_{32} - a_{42})x_2 + x_3 + x_4 \ .$$

But let us suppose for the sake of this discussion that $a_{32} - a_{42} \leq 0$.

3. We then consider increasing the value of t_1 (assuming $a_{31} - a_{41} = \Pi_1 > 0$). Increasing t_1 as far as possible (or making it basic as in the simplex method) gives the following adjusted cost form

$$(12) \quad x_4^* = x_4 + x_3 + \Pi_1 x_1 = (\Pi_1 b_1 + b_3 - b_4) + (\Pi_1 a_{10} + a_{30} - a_{40})t_0 - x_1 + (a_{32} - a_{42})t_2 - x_3 \ .$$

Let us now assume that $\Pi_1 b_1 + b_3 - b_4 > 0$ that implies that giving to t_1 the value b_1. This is sufficient to make x_4 reach feasibility.

But if we examine the relation (11) we see that it suffices to give to t_1 the value

$$-\left[\frac{(b_3 - b_4)}{a_{31} - a_{41}}\right] = t_1^o$$

to make x_4 feasible.

It is easy to see that the value t_1^o is less than or equal to the value assigned to t_1 basic. There t_1^o is the minimum value for t_1 so that there exist a solution to our problem with $t_0 = 0$; this will assure us that

$$t_0 = 0\ ,\quad t_1 = t_1^o\ ,\quad t_2 = t_2^o\ ,\quad t_3 = t_3^o \text{ (to fix)}$$

is the lexico-min vector t which gives the solution of the problem, $t_2^o = 0$ because the coefficient of t_2 was never positive in the successive "adjusted cost forms."

In order to determine the value of t_3 we fix $t_0 = 0$, $t_1 = t_1^o$, $t_2 = t_2^o = 0$ and modify our constant column accordingly.

$$\bar{b}_3 = b_3 + a_{30} \cdot 0 + a_{31} t_1^o + a_{32} \cdot 0$$

$$\bar{b}_4 = -b_4 - a_{40} \cdot 0 - a_{41} t_1^o + a_{42} \cdot 0$$

Repeating the same operations as described above, it is not difficult to show that $t_3^o = -\bar{b}_4$. This method is illustrated in Appendix B by an example.

Properties of the Deepest Cut Problem

1. The procedure suggested to solve the deepest cut problem is exactly the procedure used in the basic algorithm for generating cuts.

2. The procedure gives the best cut for the equations of the system, i.e., that given the criteria used for generating efficient cuts in the basic algorithm, there is no better cut possible.

3. The technique leads in a sequence of k steps to the solution. Each component of the vector t is considered once and only once to enter the basis, i.e., if at iteration $\nu(\leq k+j+1)$ the variable t_j was not selected for entering the basis, then for all the following iterations $\nu+1, \nu+2, \ldots$ it will never be considered again.

In the case when $k=3$, we have seen that if at iteration $\nu+2$, t_2 has a negative coefficient, this coefficient always remains negative during the solution process, t_2 is never a candidate for entering the basis.

4. It is possible to give an upper bound for the number of steps, i.e., each variable t is considered only once for entering in the basis so the number of iterations depends only on the size of the system and is in fact at most k.

2. The Deep Cut Algorithm.

If rather than dealing with the basic canonical form we had the following revised form:

Find non-negative integers x_j and t_i and maximize x_0 subject to

$$
\begin{aligned}
x_0 &= b_0 - t_0 \\
x_1 &= b_1 + t_0 - t_1 \\
&\vdots \\
x_{k+1} &= b_{k+1} - t_k - t_{k+1} \\
x_{k+2} &= b_{k+2} + a_{k+2;0}\,t_0 + a_{k+2,1}t_1 + \cdots + a_{k+2,k+1}t_{k+1} - t_{k+2} \\
&\vdots \\
x_n &= b_n + a_{n0}\,t_0 + a_{n,1}\,t_1 + \cdots + a_{n,k+1}\,t_{k+1} - t_{k+2} \\
&\vdots \\
x_{n+3} &= b_{n+3} + a_{n+3,0}t_0 + a_{n+3,1}t_1 + \cdots + a_{n+3,k+1}t_{k+1} - t_{k+2} \\
x_{n+4} &= -b_{n+4} - a_{n+4,0}t_0 - a_{n+4,1}t_1 + \cdots - a_{n+4,k+1}t_{k+1} + t_{k+2}
\end{aligned}
\tag{13}
$$

where all the a_{ij} and b_i are positive (except b_0), then a natural algorithm, using the results of the last section, suggests itself.

In order to obtain the revised form, starting from the basic canonical form (8), we subtract from the right-hand side of the equations

$$\begin{aligned} x_{k+2} &= b_{k+2} + a_{k+2,0}t_0 + a_{k+2,1}t_1 + \cdots\cdots + a_{k+2,k+1}t_{k+1} \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \\ x_{n+2} &= b_{n+2} + a_{n+2,0}t_0 + a_{n+2,1}t_1 + \cdots\cdots + a_{n+2,k+1}t_{k+1} \end{aligned}$$

a parameter t_{k+2}, and we add two equations

$$\begin{aligned} x_{n+3} &= \qquad\qquad\qquad\qquad\qquad\qquad - t_{k+2} \\ x_{n+4} &= \qquad\qquad\qquad\qquad\qquad\qquad + t_{k+2} \end{aligned}$$

which have the property that t_{k+2}, x_{n+3}, x_{n+4} must all be equal to zero in any feasible solution.

We then perform column operations in order to make the coefficient of $t_0, t_1, t_2, \ldots, t_{k+1}$ positive (except for the coefficients which will appear in the last row $n+4$). The equation generating the column operations has the form

$$(14) \quad s = -\beta_0 t_0 - \beta_1 t_1 - \beta_2 t_2 \cdots - \beta_j t_j + \cdots - \beta_{k+1} t_{k+1} + t_{k+2}$$

where $\beta_j = -[\min_i \{a_{ij}\}]$ $i = k+2, k+3, \ldots, n+2$, $j = 1, 2, 3, \ldots, k+1$.

To fix β_0 we need some more information. We want the coefficients of t_0 to be positive, but we want also to determine the coefficients of t_0 so that when we substitute $K_0 + t_0$ for t_0 (where K_0 is an admissible value for t_0) the constant terms become non-negative.

We know (when dealing with a maximization problem) that the optimal solution $(x_{0\ LP}^{o})$ of the linear program is always greater than or equal to the optimal solution $(x_{0\ IP}^{o})$ of the same problem but where integral values have to be assigned to the variables.

$$x_{0\ LP}^{o} \geq x_{0\ IP}^{o}$$

but also

$$\left[x_{0\ LP}^{o}\right] \geq x_{0\ IP}^{o}$$

let $K_0 = -\left[x_{0\ LP}^{o}\right] + b_0$ (initial b_0)

we have

$$+ x_{0\ IP}^{o} = b_0 - t_0^{o}$$

$$-\left[x_{0\ LP}^{o}\right] \leq - b_0 + t_0^{o}$$

$$t_0^{o} \geq K_0$$

so $t_0^{o} = K_0 + t_0'$. This means that the substitution $K_0 + t_0'$ for t_0 is a legitimate operation.

To choose β_0 we will assure ourselves that the coefficients of t_0 become non-negative and also that when we perform the substitution (15), described above, the constant term column will contain all non-negative components (except the last one).

$$\beta_0 = \max \left\{ - \min_i a_{i0} \ , \ \max_i \left[-\frac{b_i}{k_0} \right] - a_{i0} \right\}$$

$$i = k+2, \ldots, n+4$$

After performing the column operation generated by (14), i.e., pivoting on t_{k+2} we then add the cut

$$s = -K_0 + t_0$$

and pivot on t_0.

This gives us the tableau (13), which is equivalent to our initial problem with respect to the integer valued solution.

The Algorithm

The underlying procedure is equivalent to the basic algorithm method but in this case there is only one negative constant term, b_{n+4}. Each "subproblem" will contain

1) the last row: n+4
2) one row of the set $k+2, \ldots, n+2$ (the row $n+3$ is the opposite in sign of the row $n+4$)
3) the diagonal part of the tableau.

This is exactly the structure of the deepest cut problem.

To generate the best cut for each sub-problem we can use the method of the deepest cut problem.

And among those cuts select the best one, using the same criterion as for the basic algorithm.

This algorithm can be used if, (1) the general problem can be brought into the necessary form, and (2) it can be maintained in this form throughout the pivot steps. Both operations can be performed although the maintenance of non-negative coefficients below the diagonal does involve some number of column operations. If we omit the maintenance of the non-negative coefficient, we have the basic algorithm discussed in the first part of the paper. In that case, we generate deep cuts but not the deepest cuts for the subproblems.

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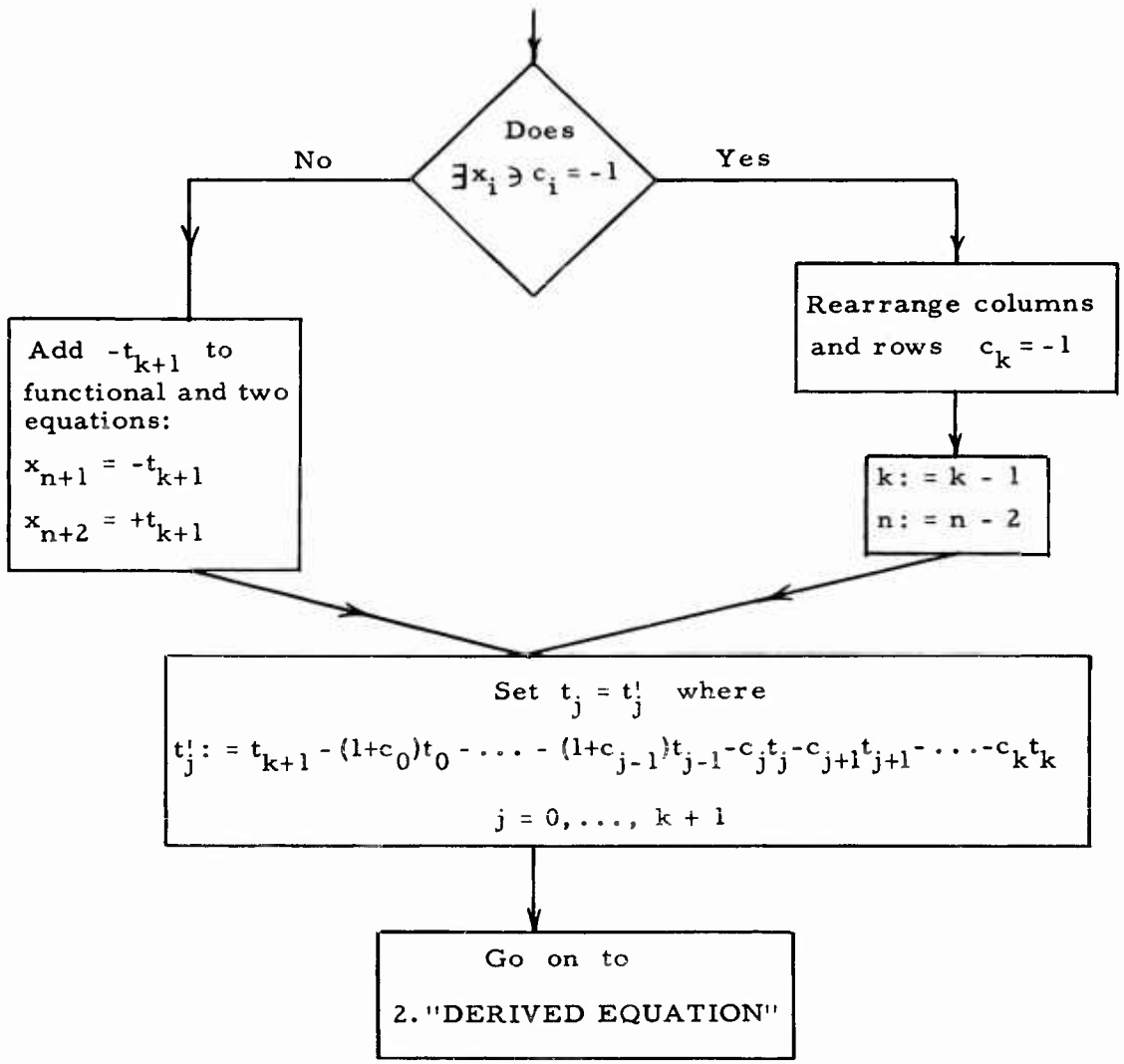
1. Dantzig, G. B., L.R. Ford, Jr., and D.R. Fulkerson, "A Primal-Dual Algorithm for Linear Programs," Annals Study 38, "Linear Inequalities and Related Systems," Kuhn and Tucker, Eds., Princeton University Press, 1956.

2. Gomory, Ralph, "All-Integer Integer Programming Algorithm," IBM Research Report RC-189, January 1960.

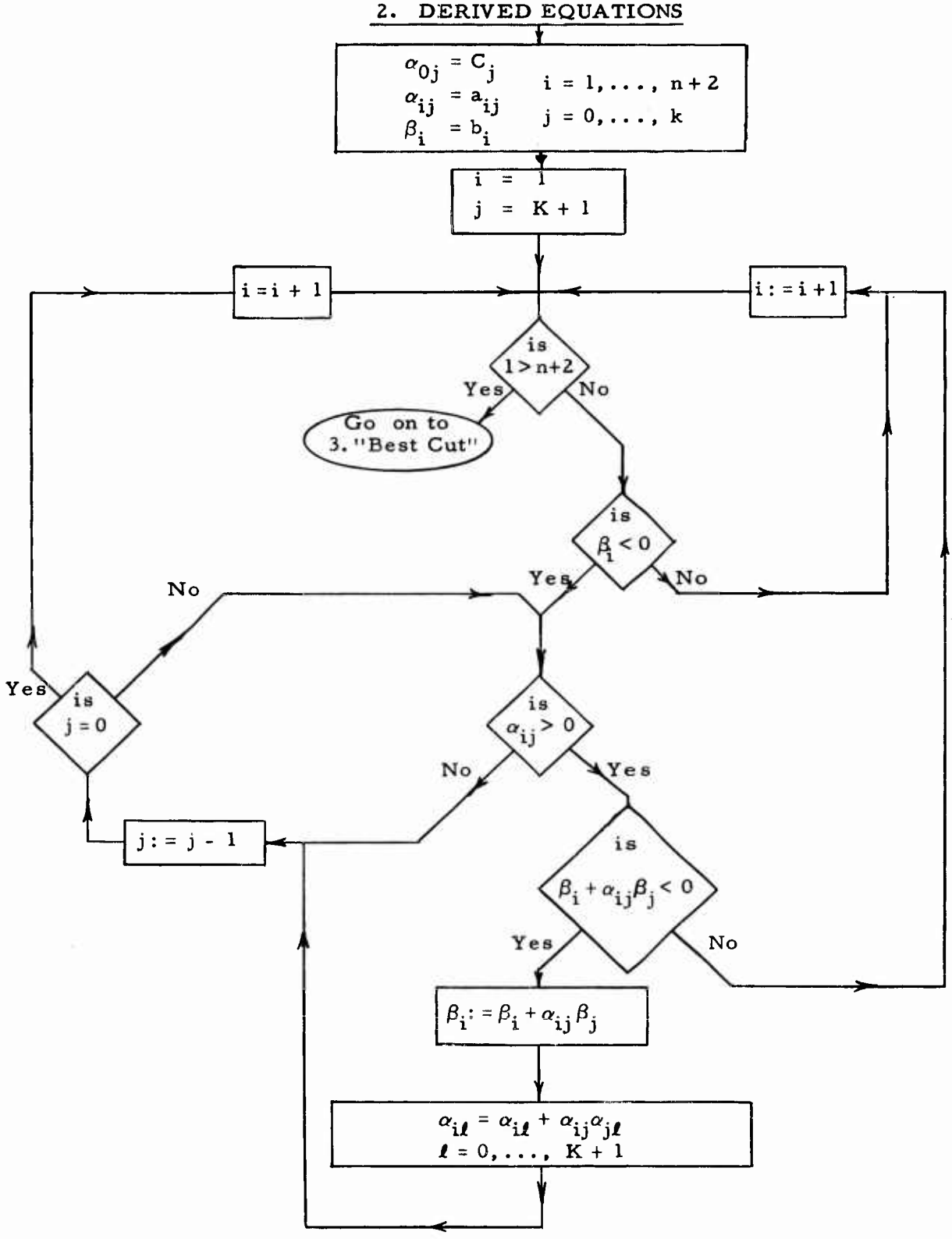
APPENDIX A

In the following four figures a flow diagram for the basic algorithm is given. Figure 1 tells us how to get the original problem in the form (1) into (5). In Figure 2 we associate to each equation with a negative constant term a linear combination of equations in which the coefficients of the rightmost t's have been made non-positive. In Figure 3 we select the linear combination for which the last positive term is furthest from the left, and in Figure 4 we actually perform the "cut" associated with this equation. The algorithm consists of repeating steps 2-4 until all the constant terms are non-negative.

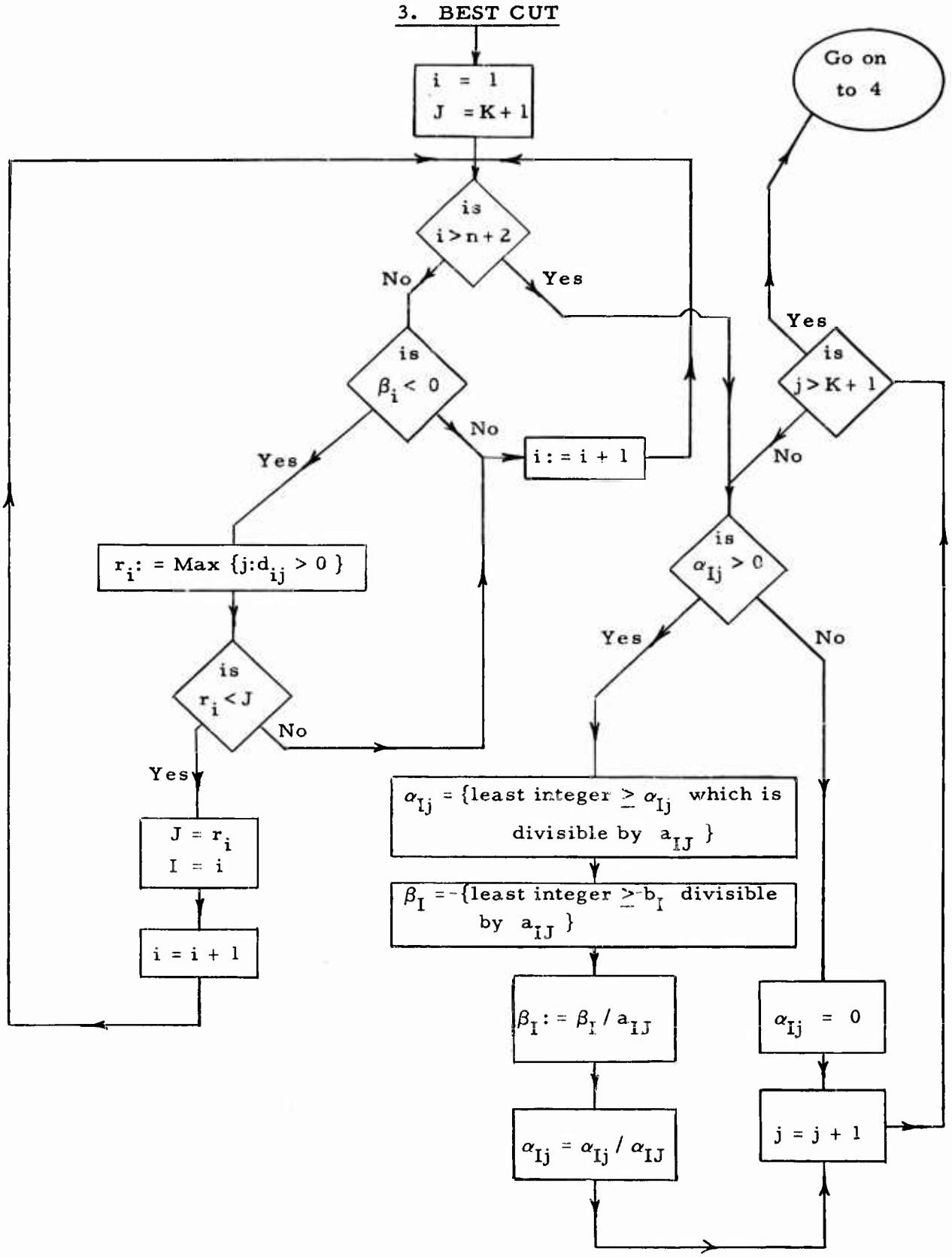
1. CANONICAL FORM



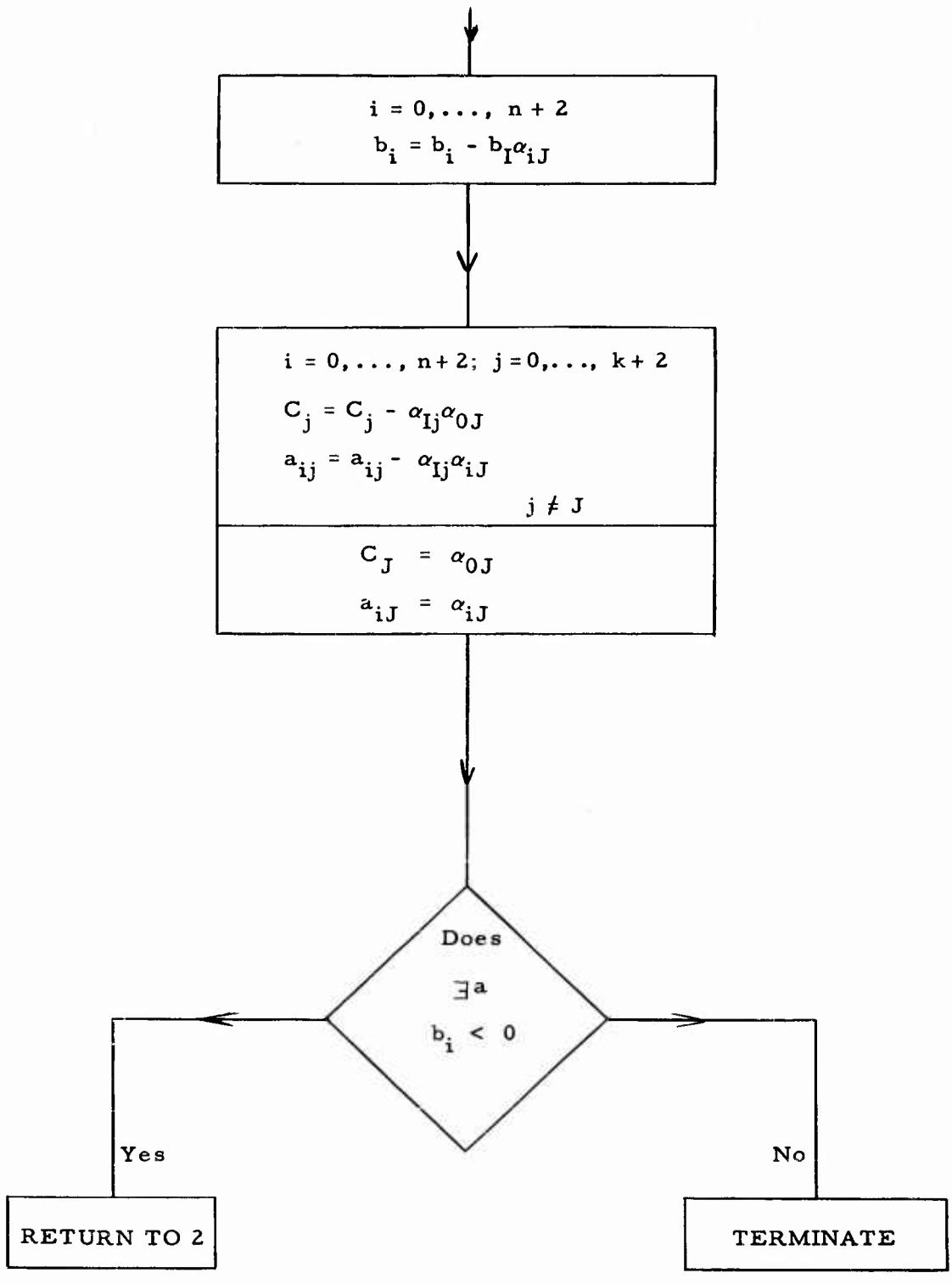
2. DERIVED EQUATIONS



3. BEST CUT


i = 1
J = K+1
is i > n+2
No
Yes
is $\beta_i < 0$
No
Yes
i := i + 1
r_i := Max {j:d_{ij} > 0 }
is r_i < J
No
Yes
J = r_i
I = i
i = i + 1
Go on to 4
Yes
is j > K+1
No
is α_{Ij} > 0
Yes
No
α_{Ij} = {least integer ≥ α_{Ij} which is divisible by a_{IJ} }
β_I = {least integer ≥ b_I divisible by a_{IJ} }
β_I := β_I / a_{IJ}
α_{Ij} = α_{Ij} / α_{IJ}
α_{Ij} = 0
j = j + 1

4. SUBSTITUTION



APPENDIX B

Basic Algorithm:

The original problem

$$\begin{aligned}
\text{Max } x_0 &= 10 - 2t_0 - t_1 - t_2 \\
x_1 &= t_0 \\
x_2 &= t_1 \\
x_3 &= t_2 \\
x_4 &= 26 + t_0 - 3t_1 - 5t_2 \\
x_5 &= -12 + t_0 + t_1 + t_2
\end{aligned}$$

Normally we use special methods to generate $a - 1$ in the objective form but in this case this is not necessary.

Make the substitutions

$$\begin{aligned}
t_0^{new} &= t_2 + 2t_0 + t_1 \\
t_1^{new} &= t_2 + t_0 + t_1 \\
t_2^{new} &= t_2 + t_0
\end{aligned}$$

we then obtain

$$\begin{aligned}
x_0 &= 10 - t_0 \\
x_1 &= t_0 - t_1 \\
x_2 &= t_1 - t_2 \\
x_3 &= -t_0 + t_1 + t_2 \\
x_4 &= 26 + 6t_0 - 9t_1 - 2t_2 \\
x_5 &= -12 + t_1
\end{aligned}$$

$$(S_1 = -15 + t_0)$$

To simplify the problem we make use of a convenient device. If we solve the problem without the integer constraints we get Max $x_{0_{LP}} = 9/12$.

Since $x_{0_{IP}}$ the integer programming solution is maximized over a more restricted set Max $x_{0_{IP}} \leq$ Max $x_{0_{LP}}$. Therefore, Max $x_{0_{IP}} \leq -5$.

Since $x_0 = 10 - t_0$ to $\geq +15$. This allows us to introduce the cut $S_1 = -15 + t_0$. Doing the appropriate pivot (essentially $t_0 := t_0 - 15$) we obtain:

$$\begin{aligned}
\text{Max} \quad x_0 &= -5 - t_0 \\
x_1 &= 15 + t_0 - t_1 \\
x_2 &= t_1 - t_2 \\
x_3 &= -15 - t_0 + t_1 + t_2 \\
x_4 &= 116 + 6t_0 - 9t_1 - 2t_2 \\
x_5 &= -12 + t_1
\end{aligned}$$

We now start using the procedure of the basic algorithm. Beginning with

$$x_3 = -15 - t_0 + t_1 + t_2$$

we make the rightmost positive coefficient zero by adding x_2

$$x_2 + x_3 = -15 - t_0 + 2t_1$$

If we try to eliminate the t_1 coefficient by adding $2x_1$ we obtain a non-negative constant term. The last equation yields the cut

$$S_2' = -8 + t_1 \Longleftrightarrow t_1 \geq S_1 .$$

Using $x_5 = -12 + t_1$ we try to eliminate t_1 but again the constant term goes non-negative so the best we can do is $S_2 = -12 + t_1$. The latter cut is better so we introduce $S_2 = -12 + t_1$. Completing the standard pivot operations our tableau becomes

$$\begin{aligned} \text{Max} \quad x_0 &= -5 - t_0 \\ x_1 &= 3 + t_0 - t_1 \\ x_2 &= 12 \qquad\quad t_1 - t_2 \\ x_3 &= -3 - t_0 + t_1 + t_2 \\ x_4 &= 8 + 6t_0 - 9t_1 - 2t_2 \\ x_5 &= \qquad\quad t_1 \end{aligned}$$

here we have only one equation with a negative constant term which implies the cut $S_3 = -3 + t_1 + t_2$ making the substitution we obtain the optimal solution

$$\begin{aligned} x_0 &= -5 - t_0 \\ x_1 &= 3 + t_0 - t_1 \\ x_2 &+ 9 \qquad + 2t_1 - t_2 \\ x_3 &+ 0 - t_0 \qquad + t_2 \\ x_4 &= 2 + 6t_0 - 7t_1 - 2t_2 \\ x_5 &= 0 \qquad + t_1 \end{aligned}$$

If we did not use the linear programming solution to generate the first cut this cut would have been

$$S_1 = -12 + t_0$$

derived from the following linear combination: $x_1 + x_5$. The tableau then becomes:

$$
\begin{aligned}
x_0 &= -2 - t_0 \\
x_1 &= 12 + t_0 - t_1 \\
x_2 &= \qquad\quad t_1 - t_2 \\
x_3 &= -12 - t_0 + t_1 + t_2 \\
x_4 &= 98 + 6t_0 - 9t_1 - 2t_2 \\
x_5 &= -12 \qquad + t_1
\end{aligned}
$$

our best following cut is: $S_2 = -12 + t_1$ derived from x_5.

It is not difficult to verify that no better cut is available. We then obtain:

$$
\begin{aligned}
x_0 &= -2 - t_0 \\
x_1 &= \qquad t_0 - t_1 \\
x_2 &= 12 \qquad\quad t_1 - t_2 \\
x_3 &= \quad - t_0 + t_1 + t_2 \\
x_4 &= -10 + 6t_0 - gt_1 - 2t_2 \\
x_5 &= \qquad\quad + t_1
\end{aligned}
$$

We then get the following cuts

$$
\begin{aligned}
s_3 &= -2 + t_0 \\
s_4 &= -2 \qquad + t_1 + t_2 \\
s_5 &= -1 + t_0 \\
s_6 &= -1 \qquad + t_1 + t_2
\end{aligned}
$$

and obtain the same final tableau. It is interesting to see that this required 6 cuts where the previous method only required 3 cuts.

APPENDIX C

The Deepest Cut Problem:

Find non-negative integers x_i $(i = 1, 2, 3, 4)$, t_j $(j = 1, 2, 3)$, and maximize x_0 subject to:

$$x_0 = 3 - t_0$$

$$x_1 = 1 + t_0 - t_1$$

$$x_2 = 3 + 2t_0 + 2t_1 - t_2$$

$$x_3 = 8 + 3t_1 + 6t_2 - t_3$$

"OBJ" $$x_4 = -26 - t_0 - 12t_1 - 3t_2 + t_3$$

Basic variables, x_0, x_1, x_2, x_3.

(a) We first ask is there a solution with $t_0, t_1, t_2 = 0$? Making t_3 basic and increasing as far as we can to $t_3 = 8$ we get $x_4 = -18$. Eliminating t_3 from "OBJ" we obtain

$$x_4 + x_3 = -18 - t_0 - 9t_1 + 3t_2 \quad .$$

(b) Since x_4 with $t_0, t_1, t_2 = 0$ is $= -18$ we now allow t_2 to become positive and require only that $t_0, t_1 = 0$. Making t_2 basic and eliminating from the "adjusted OBJ" we get

$$x_4 + x_3 + 3x_2 = -9 + 5t_0 - 3t_1 \, .$$

(c) Since the coefficient of t_1 is negative in the current "OBJ" introducing t_1 will only make the problem more infeasible.

(d) Finally, by making $t_0 \geq 2$ we can make the problem feasible. Since we want the lexico-minimum vector t we take $t_0 = 2$.

The values of the rest of the variables can be determined whether by back substitution or by repeating the procedure with t_0 no longer a variable but fixed at 2, e.g., next try to make x_4 feasible in

$$x_0 = 1$$
$$x_1 = 3 - t_1$$
$$x_2 = 9 + 2t_1 - t_2$$
$$x_3 = 8 + 3t_1 + 6t_2 - t_3$$
$$x_4 = -28 - 12t_1 - 3t_2 - t_3$$

Repeating the same process, we can fix t_1; then t_2, t_1 being fixed, and so on.

The complete solution is

$x_0 = 1$ $x_1 = 3$ $x_2 = 0$ $x_3 = 7$ $x_4 = 0$

$t_0 = 2$ $t_1 = 0$ $t_2 = 9$ $t_3 = 55$.

Practically one can affect each row of a multiplier

$$\pi_4^4 = 1(\text{OBJ}), \quad \pi_3^4 = 1. \qquad \pi_2^4 = (a_{32} - a_{42}) = (6-3) = 3.$$

$$\pi_1^4 = \text{Max}\left\{(-12 + 3 + 3\pi_2^4) = -3,\ 0\right\} = 0.$$

which remains always the same and may be considered to some extent as playing an identical rule to the simplex multipliers.

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